# ENGINEERING PHYSICS

# UNIT - 1

# INTRODUCTION TO QUANTUM MECHANICS

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References: VR Sunitha's Lectures
Vishrutn V's notes

Vibha Masti

# Electric & Magnetic fields

## Vector Fields

- Wind, fluids
- Gravitational field
- Electric & Magnetic field
- Represented by vectors in space

## Vector Operator (del)

- del operator – $\nabla$
- $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$
- vector operator with no magnitude
- partial differential operator
- operator acts on vector field as either cross or dot product
- $\vec{F}(x,y) = \hat{i}x^2 + \hat{j}y^2$
- When $\nabla$ operates on scalar field: gradient
- $\nabla\phi$ where $\phi(x,y,z)$ is a scalar field is the gradient
- Gradient gives direction along which steepest change of field occurs
- Gradient of a scalar is a vector

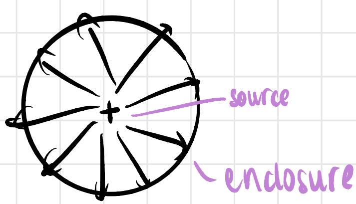
## Divergence ($\nabla \cdot \vec{F}$)

- Divergence means flow
- If $\vec{\nabla}\cdot\vec{F}$ = +ve : flow is outward
  If $\vec{\nabla}\cdot\vec{F}$ = -ve : flow is inward
  If $\vec{\nabla}\cdot\vec{F} = 0$ : inward flow = outward flow

at a given point

## Electric Fields

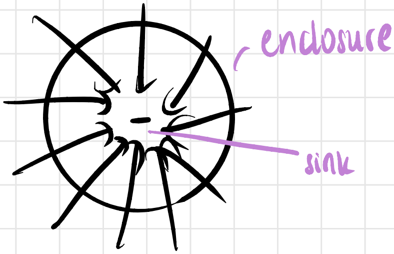
1) isolated +ve charge




Divergence - flow

$\vec{\nabla}\cdot$ : +ve

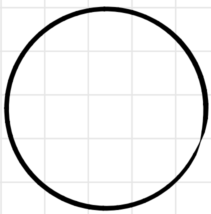
2) isolated -ve charge




Divergence - flow

$\vec{\nabla}\cdot$ : -ve

3) Free space

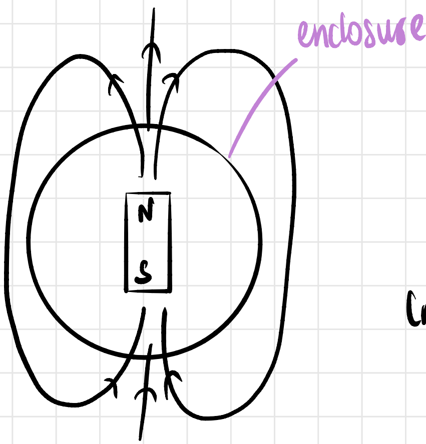


no flow

$\vec{\nabla}\cdot = 0$

## magnetic fields

1) Magnetic dipole




no flow

$\vec{\nabla}\cdot = 0$

(no magnetic monopole)

$$\vec{F}(x,y) = \hat{i}x^2 + \hat{j}y^2$$

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}x^2 + \hat{j}y^2)$$

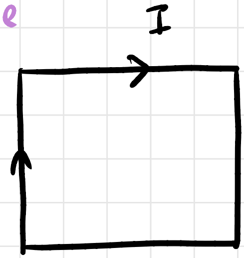
$$\vec{\nabla}\cdot\vec{F} = 2x + 2y$$

Divergence of a vector field gives a scalar function.

## curl ($\nabla \times \vec{F}$)

- rotation of vector fields
- whirlpools, tornado, ocean current, centrifuges
- can be clockwise or anticlockwise

1) Purely resistive circuit




J = current density

$\vec{\nabla} \cdot J = 0$ ; no flow

There is curl

$\vec{\nabla} \times J \neq 0$

## Laplacian Operator ($\nabla^2 = \nabla \cdot \nabla$)

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

# MAXWELL'S EQUATIONS

## INTEGRAL FORM

1) $$\int_s \vec{E} \cdot d\vec{s} = \frac{q}{\varepsilon_0}$$

2) $$\int_s \vec{B} \cdot d\vec{s} = 0$$

Gauss' Law

(larger regions of space)

3) $$\int_l \vec{E} \cdot d\vec{l} = -\frac{d\phi}{dt}$$

Faraday's Law

4) $$\int_l \vec{B} \cdot d\vec{l} = \mu_0 (I_D + I)$$

modified Ampére's Law

## DIFFERENTIAL FORM

1) $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

2) $$\vec{\nabla} \cdot \vec{B} = 0$$

Gauss' Law

(smaller regions of space)

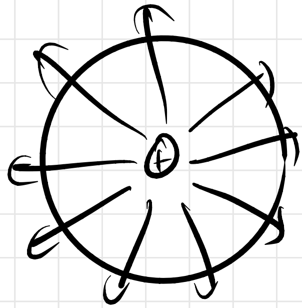
3) $$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

Faraday's Law

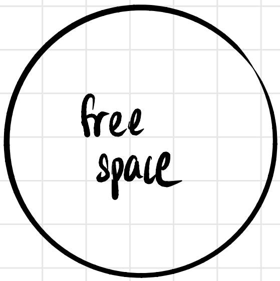
4) $$\vec{\nabla} \times \vec{B} = \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right)$$

modified Ampére's Law

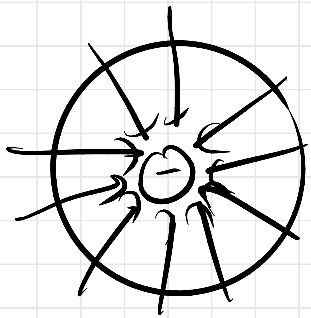
1) $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$




$\nabla \cdot E = +ve$

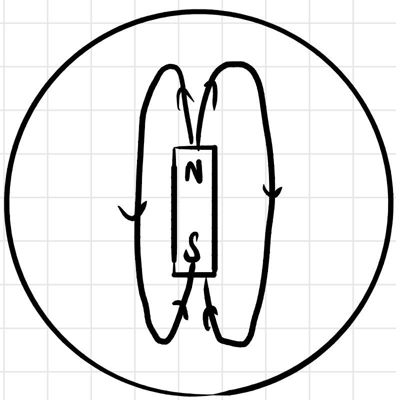



$\nabla \cdot E = 0$




$\nabla \cdot E = -ve$

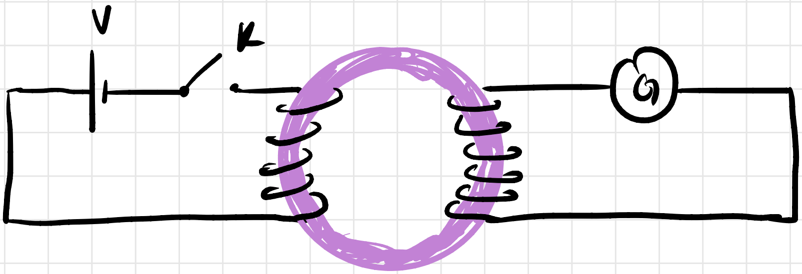
2) $\nabla \cdot B = 0$




$\nabla \cdot B = 0$

magnetic monopoles do not exist

3) $\nabla \times E = \frac{-\partial B}{\partial t}$




- circulating $\vec{E}$ can give rise to time-varying $\vec{B}$
- time-varying $\vec{B}$ gives rise to circulating $\vec{E}$

4) $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

## Ampere's Law

$$\nabla \times H = J$$

## IDENTITIES

scalar triple product

1. $\nabla \cdot (\nabla \times A) = 0$

vector triple product (Lagrange's formula)

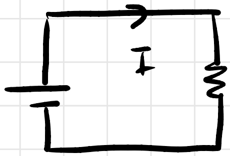
2. $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A \cdot C) B - (A \cdot B) C$$

$$\nabla \times H = J$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J$$

## When only resistive element

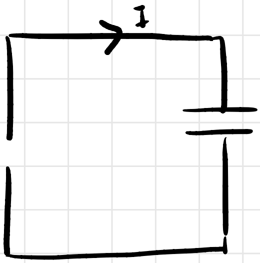



$$J = I/A$$

$$\nabla \cdot J = 0$$

$$\nabla \cdot (\nabla \times H) = 0$$

## With capacitor




$$\nabla \cdot J \neq 0$$

$\vec{E}$ is varying

$$\nabla \cdot (\nabla \times H) = 0$$

## Maxwell's Correction

$$\nabla \times H = J + J_D$$

$$D = \varepsilon_0 E$$

$$J_D = \frac{\partial D}{\partial t} \Rightarrow J_D = \varepsilon_0 \frac{\partial E}{\partial t}$$

$$B = \mu_0 H \Rightarrow H = \frac{B}{\mu_0}$$

$$\nabla \times \frac{B}{\mu_0} = J + \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

# MAXWELL'S EQUATIONS IN FREE SPACE

no charge, no current ($\rho = 0$, $J = 0$)

1) $\vec{\nabla} \cdot \vec{E} = 0$

2) $\vec{\nabla} \cdot \vec{B} = 0$

3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

4) $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

## Using Maxwell's Equations in Free Space, Derive Wave Equation in terms of Electric Field and Magnetic Field.

### ELECTRIC FIELD

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$\vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = 0 - \nabla^2 \vec{E}$$

$$\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t}\right) = \nabla^2 \vec{E}$$

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = \nabla^2 \vec{E}$$

$$\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \nabla^2 \vec{E}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \quad \text{——— (1)}$$

General wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{——— (2)}$$

where v is the velocity of propagation of the wave

Expanding (1), we get

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{——— (3)}$$

Comparing 2 and 3, we get

wave equation in terms of $\vec{E}$

$$\boxed{\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

where $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$\vec{E}$ propagate through free space at the speed of light.

MAGNETIC FIELD

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$$

$$\vec{\nabla} \times \left( \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0 - \nabla^2 \vec{B}$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( \vec{\nabla} \times \vec{E} \right) = -\nabla^2 \vec{B}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\nabla^2 \vec{B}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B}$$

$$\boxed{\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}}$$

wave equation in terms of $\vec{B}$

where $c^2 = \dfrac{1}{\mu_0 \varepsilon_0}$

magnetic fields propagate through free space at the speed of light.

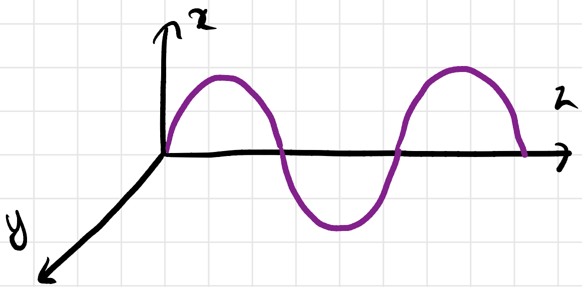
Varying $\vec{E}$ and $\vec{B}$ represent light waves.

Light waves are electromagnetic waves.

Show that $\vec{E}$ and $\vec{B}$ are perpendicular to each other and to the direction of propagation.

ELECTRIC WAVE

$$E_x = E_{0x} \cos(\omega t - kz)$$




$$E = \hat{i} E_x + \hat{j} E_y + k E_z$$

let us consider only the x-component of $\vec{E}$

$\therefore$ $E_y = 0$, $E_z = 0$

(Polarised wave)

$$\nabla \times E = \frac{-\partial B}{\partial t}$$

$$\nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$\nabla \times E = \hat{j}\left(\frac{\partial E_x}{\partial z}\right) - \hat{k}\left(\frac{\partial E_x}{\partial y}\right)$$

$E_x$ is independent of $y$

$$\nabla \times E = \hat{j} \frac{\partial E_x}{\partial z}$$

$$\nabla \times E = \hat{j} \frac{\partial}{\partial z}\left(E_{0x} \cos(\omega t - kz)\right)$$

$$= \hat{j} \, (-k)(-)E_{0x} \sin(\omega t - kz)$$

$$\nabla \times E = \hat{j} \, E_{0x} k \sin(\omega t - kz)$$

$$\frac{-\partial B}{\partial t} = \hat{j} \, E_{0x} k \sin(\omega t - kz)$$

Integrate wrt t

$$-B = \hat{j}\, E_{0x} k \left( \frac{-\cos(\omega t - kz)}{\omega} \right)$$

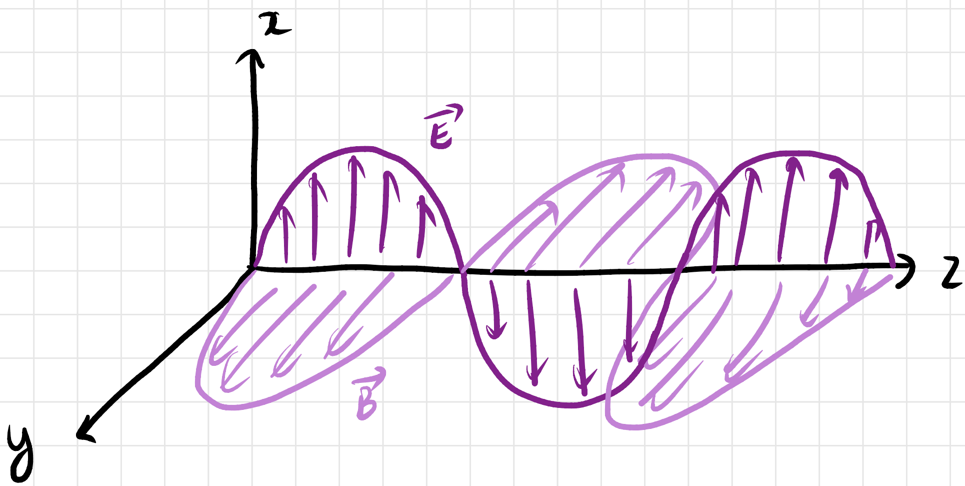
$$B = \hat{j}\, \frac{k}{\omega} E_{0x} \cos(\omega t - kz)$$

$$\omega = 2\pi f = \frac{2\pi c}{\lambda} = kc \Rightarrow \frac{k}{\omega} = \frac{1}{c}$$

$$B = \hat{j}\, \frac{E_x}{c}$$

∴ Magnetic field is along y-direction

$$B_y = \frac{E_x}{c}$$




$\vec{E}$ and $\vec{B}$ are always coupled to each other and cannot be isolated.

# Properties of EM Waves

1. $E \perp B$
2. $E$ & $B$ $\perp$ direction of propagation
3. $E$ & $D \longrightarrow$ speed of light
4. EM waves carry energy

## ENERGY DENSITY

### Energy carried by electric field

$$\text{energy density of } \vec{E} = \frac{1}{2}\varepsilon_0 E^2$$

### Energy carried by magnetic field

$$\text{energy density of } \vec{B} = \frac{1}{2\mu_0} B^2$$

### Total energy density

$$U_T = \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\frac{B_y^2}{\mu_0}$$

$$B_y = \frac{E_x}{c}$$

$$U_T = \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\frac{E_x^2}{c^2\mu_0}$$

$$\frac{1}{c^2} = \mu_0\varepsilon_0$$

$$U_T = \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\varepsilon_0 E_x^2$$

$$U_T = \varepsilon_0 E_x^2 \quad \text{or} \quad U_T = \frac{B_y^2}{\mu_0}$$

# POYNTING VECTOR

Amount of energy flowing through EM waves per unit area per unit time.

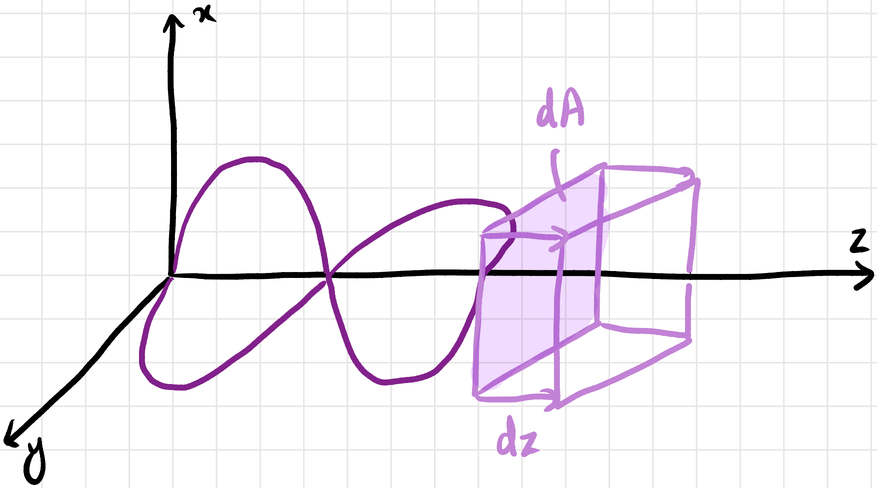
$$S = \frac{du}{dA\,dt}$$

$\frac{\text{power}}{\text{area}}$ = intensity

$$\text{units} = \frac{\overset{W}{J}}{m^2 s} = W m^{-2}$$

Consider a polarised EM wave propagating in space

Over a time $dt$, the wave moves from $z$ to $z+dz$




$$dV = dA \cdot dz$$
$$= dA \cdot c \cdot dt$$

$$du = U_T \, dV$$

$$du = \varepsilon_0 E_x^2 \, dA \, c \, dt \qquad E_x = c B_y$$

$$du = \varepsilon_0 E_x \, c B_y \, c \, dA \, dt$$
$$= \varepsilon_0 E_x B_y \, c^2 \, dA \, dt \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$du = \frac{E_x B_y}{\mu_0} \, dA \, dt$$

$$\frac{du}{dA \, dt} = S = \frac{E_x B_y}{\mu_0}$$

We take the direction of $\vec{S}$ in the direction of propagation of the wave

∴ We take $\vec{S} \parallel \vec{E} \times \vec{B}$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

## Average Value of $\vec{S}$ ($\langle S \rangle$)

$$S = \frac{E_x B_y}{\mu_0}$$

$$= \frac{E_x \, E_x}{\mu_0 c}$$

$$\mu_0 c = \frac{1}{\varepsilon_0 c}$$

$$= E_x^2 \varepsilon_0 c$$

$$S = c \varepsilon_0 E_x^2$$

$$E_x = E_{0x} \cos(\omega t - kz)$$

$$\langle S \rangle = c \varepsilon_0 \langle E_x^2 \rangle$$

$$= c \varepsilon_0 E_{0x}^2 \langle \cos^2(\omega t - kz) \rangle$$

$$\langle \cos^2(\omega t - kz) \rangle = \frac{1}{T} \int_0^T \cos^2(\omega t - kz) \, dt = \frac{1}{2}$$

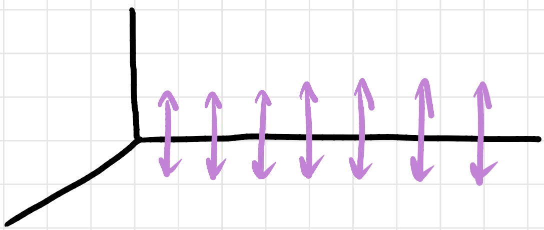
$$\boxed{\langle S \rangle = \frac{1}{2} c \varepsilon_0 E_{0x}^2}$$

- $S \propto E_{ox}^2$, where $E_{ox}$ is the amplitude
- $\therefore$ S talks about the intensity of radiation (I)
- EM waves only talk about intensity, not frequency

# POLARISATION

- Note: $\vec{E}$ of matter interacts only with $\vec{E}$ of EM wave, not $\vec{B}$
- Only in some cases (MRI scans), it interacts with $\vec{B}$

## LINEAR POLARISATION / PLANE POLARISATION



$\vec{E}$ only along x-direction

$E_x = E_{ox} \sin(\omega t - kz)$

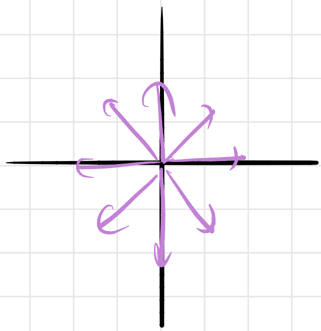
$E_y = 0$

## CIRCULAR POLARISATION

- two mutally perpendicular waves of equal amplitude with a phase difference of $\pi/2$

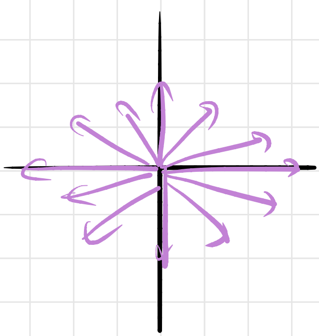
$$E_x = E_0 \sin(\omega t - kz)\,\hat{i}$$

$$E_y = E_0 \sin\left(\omega t - kz + \frac{\pi}{2}\right)\hat{j}$$



## ELLIPTICAL POLARISATION

- two mutually perpendicular waves of different amplitude with a phase difference of $\pi/2$ (right elliptical)



# DUAL NATURE OF RADIATION

## Radiation as a Wave

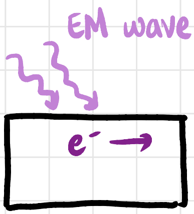
1. interference
2. diffraction
3. polarisation
4. reflection/refraction

## Radiation as Particles

1. photoelectric effect
2. blackbody radiation
3. atomic spectra
4. compton effect

## Photoelectric Effect

- Observation, experiment - Hertz




- Light incident on metals creates photo electrons
- Instantaneous emission of photoelectrons
- Wave theory could not explain this phenomenon
- Discrete bundles of energy from EM wave

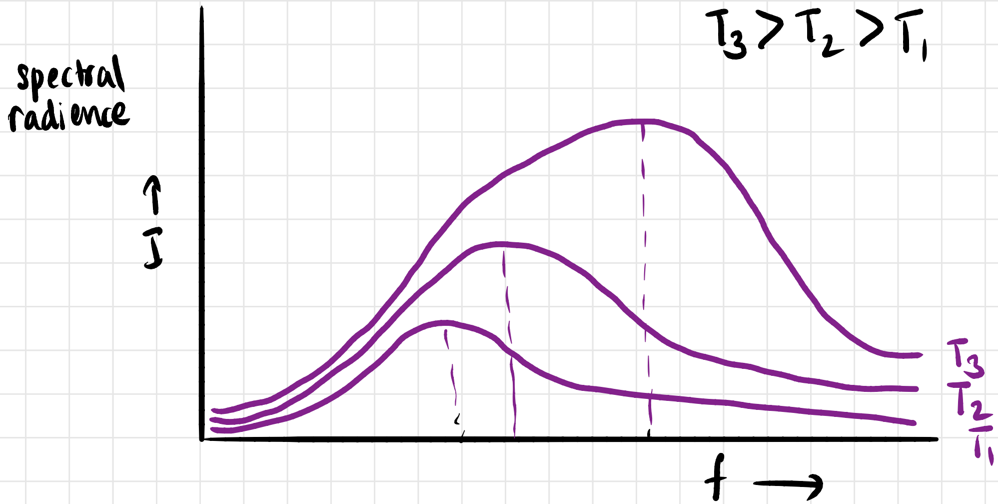
- Photon completely transfers energy to $e^-$
- Particle-particle interaction

$$h\nu = h\nu_0 + \frac{1}{2}mv^2$$

- Frequency $\propto$ energy ; intensity $\propto$ photo current

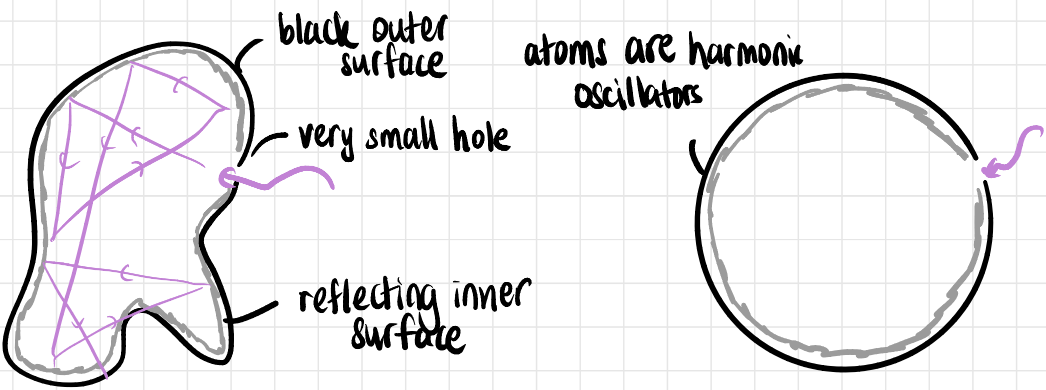
# Blackbody Radiation

- blackbodies can be used for solar cells
- spectrum: variation of intensity as a function of $\lambda$ or $f$
- experimental graph




## Blackbody

- body that completely absorbs all incident radiation
- completely emits all absorbed energy
- zero reflection
- carbon black (soot), sun



## Observations

1. As T ↑, max intensity shifts towards higher frequency

$T \propto \frac{1}{\lambda_{max}}$, $T \propto f_{max}$ Wein's displacement Law

$$E(\lambda)\,d\lambda = c_1 \lambda^{-5} e^{-c_2/\lambda T}\,d\lambda$$

2. Energy radiated is proportional to $T^4$

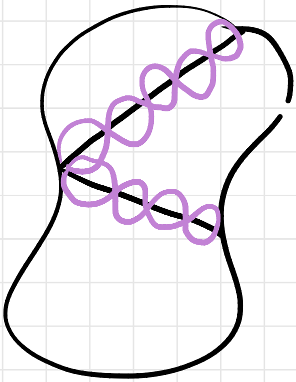
$E \propto T^4$ Stefan's Law

## Spectral Density / Spectral Radiance

the amount of energy contained in the cavity per unit volume in the interval $\nu + d\nu$ or $\lambda + d\lambda$ at a constant temperature

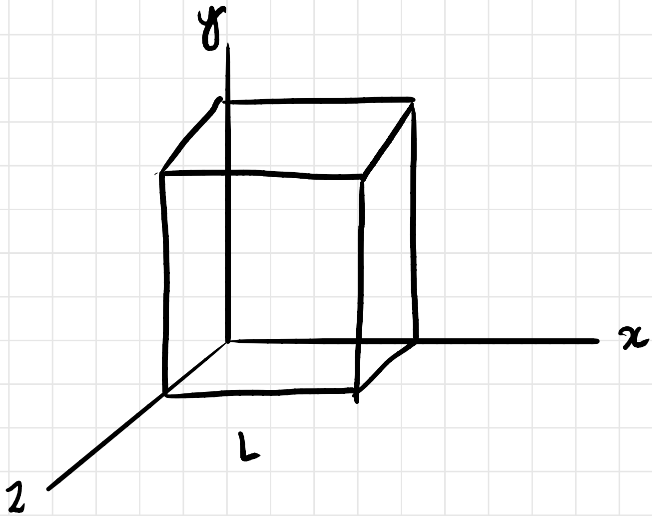
$$U_\nu\, d\nu = \frac{\text{no. of oscillators}}{\text{volume}} \times \text{average energy}$$

$$= \frac{\text{no. of standing waves}}{\text{volume}} \times \text{average energy}$$

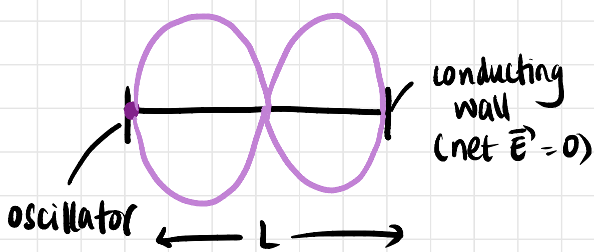


# Derivation of Rayleigh-Jeans Expression for Energy Density

They imagined EM waves in a cubic volume due to oscillating dipoles that make up the walls of the cavity




For one-dimensional cavity, only certain frequencies can stabilise to form standing waves




$$L = \frac{n\lambda}{2} \Rightarrow \lambda = \frac{2L}{n} \quad \text{or} \quad \nu = \frac{cn}{2L}$$

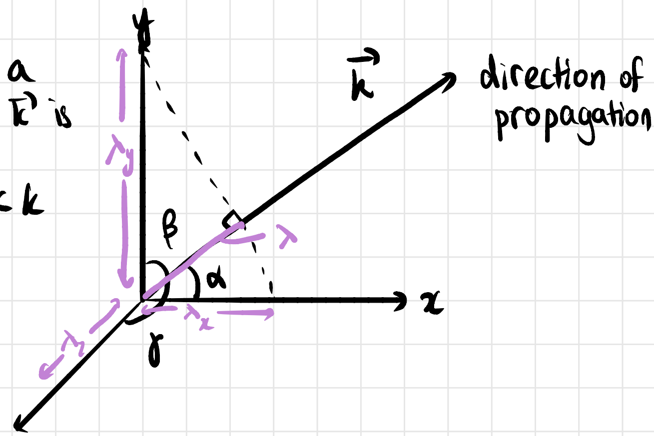
In a 3-D cubic cavity, wave can form standing wave in a direction only if each of its components independently forms standing waves in x-y-z directions

Note: $\lambda$ is not a vector, only $\vec{k}$ is

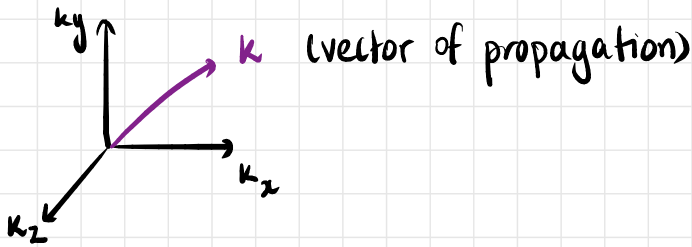
$\therefore k_x, k_y, k_z < k$

but

$\lambda_x, \lambda_y, \lambda_z > \lambda$




We get components of k in all three directions




Now, $k = \frac{2\pi}{\lambda}$ $\Rightarrow$ we get $\lambda$ (scalar) along 3 axes

For standing waves along 3 axes,

$$\lambda_x = \frac{2L}{n_x} \quad \lambda_y = \frac{2L}{n_y} \quad \lambda_z = \frac{2L}{n_z}$$

Taking $\alpha$, $\beta$ and $\gamma$ as directions,

$$\cos\alpha = \frac{\lambda}{\lambda_x}, \quad \cos\beta = \frac{\lambda}{\lambda_y}, \quad \cos\gamma = \frac{\lambda}{\lambda_z}$$

We know (dir. ratios)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\lambda^2\left(\frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} + \frac{1}{\lambda_z^2}\right) = 1$$

$$\frac{c^2}{\nu^2}\left(\frac{\nu_x^2}{c^2} + \frac{\nu_y^2}{c^2} + \frac{\nu_z^2}{c^2}\right) = 1$$

$$\nu^2 = \nu_x^2 + \nu_y^2 + \nu_z^2$$

we know $\nu = \frac{nc}{2L}$

$$\nu^2 = \frac{c^2}{4L^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$\nu = \frac{c}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

Lowest possible mode of frequency,

$$n_x = 1, \quad n_y = 1, \quad n_z = 1$$

$$\nu_{111} = \frac{c}{2L}\sqrt{3}$$

Second frequency:

Three possible modes (112, 121, 211)

$$\nu_{112} = \nu_{121} = \nu_{211} = \frac{c}{2L}\sqrt{6}$$

Modes are all the possible standing waves.

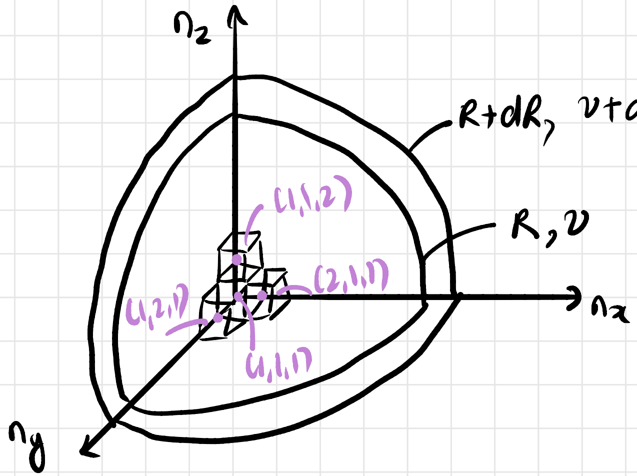
The frequencies $\nu_{112}$, $\nu_{121}$, $\nu_{211}$ are equal but are different modes, as they are physically different (different directions of propagation)

## Phase space

We plot the modes on a phase space.

We imagine an octet of a sphere to get the number of possible modes

Each possible mode can be represented as a unit cube




Volume of octet = no. of unit cubes (approx as octet is huge)

For each cube there is only 1 point which represents 1 possible mode

$\therefore$ volume = no. of unit cubes = no. of points = no. of modes

volume = no. of modes

According to the equation of a sphere $x^2+y^2+z^2=R^2$

Here $n_x^2+n_y^2+n_z^2=R^2$

$$\therefore \nu = \frac{c}{2L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{cR}{2L}$$

Let sphere be of radius $R$; all points on the surface have frequency $\nu$

No. of modes within frequency $\nu$ = volume of octet

$$= \frac{1}{8} \times \frac{4}{3}\pi R^3$$

Slightly bigger octet of radius $R+dR$ with modes within frequency $\nu + d\nu$

No. of modes within frequency $\nu + d\nu$

$$\frac{1}{8} \times \frac{4}{3}\pi (R+dR)^3$$

No. of modes with frequency lying between $\nu$ and $\nu + d\nu$

$$= \frac{1}{8} \times \frac{4\pi}{3} \left[ (R+dR)^3 - R^3 \right]$$

$$= \frac{\pi}{6} \left[ \cancel{R^3} + 3R^2 dR + \cancel{3R(dR)^2} + \cancel{(dR)^3} - \cancel{R^3} \right]$$

$dR$ is very small (neglecting higher order terms)

$$= \frac{\pi}{6} (3R^2 dR) = \frac{\pi}{2} R^2 dR$$

Now, $\nu = \frac{cR}{2L} \Rightarrow R = \frac{2L\nu}{c} \Rightarrow dR = \frac{2L}{c} d\nu$

no. of modes from $\nu$ to $\nu + d\nu$ in a unit cube $= \frac{\pi}{\cancel{2}} \left( \frac{4L^2\nu^2}{c^2} \right) \left( \frac{\cancel{2}L}{c} d\nu \right)$

$$N(\nu)\, d\nu = \frac{4\pi L^3 \nu^2}{c^3} d\nu$$

Now, Rayleigh - Jean assumed average energy per mode is $kT$ at temperature $T$ (equipartition theorem)

Energy of nodes lying between $\nu$ and $\nu + d\nu$

$$E(\nu)\,d\nu = \frac{4\pi\nu^2}{c^3} kT\,d\nu$$

$$c = \nu\lambda \Rightarrow c^2 = \nu^2\lambda^2$$

$$\nu = \frac{c}{\lambda} \Rightarrow d\nu = \frac{-c}{\lambda^2}\,d\lambda$$

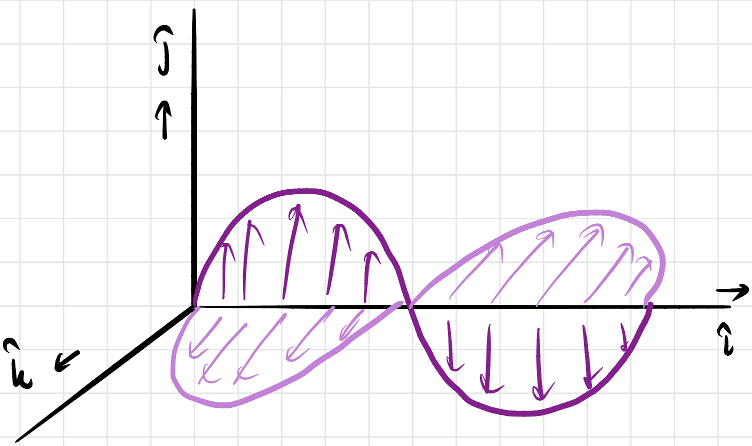
$$E(\lambda)\,d\lambda = \frac{4\pi \cancel{c^2}}{\lambda^2 \cancel{c^3}}\left(\frac{-\cancel{c}}{\lambda^2}\right) kT\,d\lambda$$

$$E(\lambda)\,d\lambda = \frac{-4\pi}{\lambda^4} kT\,d\lambda$$

energy possessed by osullators

This energy is only considering 1 direction of oscillations of waves for every direction of propagation

For direction of propagation $\hat{i}$, there are 2 orthogonal modes (along $\hat{j}$ and $\hat{k}$)




Now, no. of modes between $\nu$ and $\nu + d\nu$

$$N(\nu)d\nu = 2 \times \frac{4\pi\nu^2}{c^3}\, d\nu$$

$$= \frac{8\pi\nu^2}{c^3}\, d\nu$$

Energy per mode is $kT$

$$E(\nu)d\nu = \frac{8\pi\nu^2 kT}{c^3}\, d\nu$$

$$E(\lambda)\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda$$

# Max Planck's Theory

Due to failure of R-J, he assumed that oscillators can only oscillate at certain frequencies

fitting factor

Therefore, energy is quantised (multiples of $h\nu$)

Number of oscillators with energy $nh\nu$

$$N_n \propto e^{-\frac{nh\nu}{kT}} \quad \text{(Boltzmann equation)}$$

$$N_n = A e^{-\frac{nh\nu}{kT}}$$

Energy of $N_n$ oscillators

$$E_n = N_n n h\nu$$

Total energy

$$\sum_{n=0}^{\infty} E_n = \sum_{n=0}^{\infty} N_n n h\nu$$

Average energy

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} A n h\nu e^{-\frac{E_n}{kT}}}{\sum_{n=0}^{\infty} A e^{-\frac{E_n}{kT}}} = \frac{\sum_{n=0}^{\infty} n h\nu e^{-\frac{nh\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{nh\nu}{kT}}} = \frac{h\nu \sum_{n=0}^{\infty} n e^{-\frac{nh\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{nh\nu}{kT}}}$$

let $\alpha = \frac{h\nu}{kT}$

$$\langle E\rangle = \frac{kT\sum_{n=0}^{\infty}\alpha n e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}}$$

Writing in differential form

$$\langle E\rangle = -kT\alpha\frac{d}{d\alpha}\left(\ln\left(\sum_{n=0}^{\infty} e^{-n\alpha}\right)\right)$$

$$\sum_{n=0}^{\infty} e^{-n\alpha} = 1 + e^{-\alpha} + e^{-2\alpha} + \dots$$

GP with CR $= e^{-\alpha}$

$$\text{sum} = \frac{1}{1-e^{-\alpha}} = (1-e^{-\alpha})^{-1}$$

$$\sum_{n=0}^{\infty} e^{-n\alpha} = (1-e^{-\alpha})^{-1}$$

$$\langle E\rangle = -\alpha kT\frac{d}{d\alpha}\ln(1-e^{-\alpha})^{-1}$$

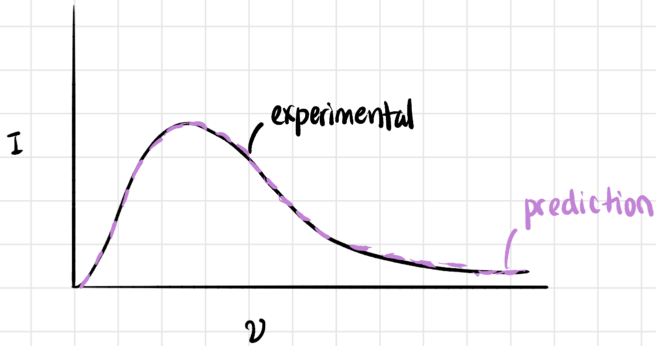
$$= \alpha kT\frac{d}{d\alpha}\ln(1-e^{-\alpha}) = \alpha kT\left(\frac{e^{-\alpha}}{1-e^{-\alpha}}\right)$$

$$= h\nu\left(\frac{1}{e^{\alpha}-1}\right) = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$ — average energy

$$U(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3}\,d\nu \quad \frac{h\nu\,d\nu}{e^{\frac{h\nu}{kT}} - 1}$$ — Max Planck's Law

$$= \frac{8\pi h\nu^3}{c^3}\left(\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right) d\nu$$




Case I:
- high frequencies
- as $h\nu >> kT$, $I \to 0$
- no ultraviolet catastrophe

Case II
- low frequencies
- $h\nu << kT$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$e^{\frac{h\nu}{kT}} = 1 + \frac{h\nu}{kT} + \frac{1}{2}\left(\frac{h\nu}{kT}\right)^2 + \ldots$$

$$e^{\frac{h\nu}{kT}} = 1 + \frac{h\nu}{kT}$$

Substituting in Max Planck's Law,

$$U(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \partial\nu \frac{h\nu}{h\nu/kT}$$

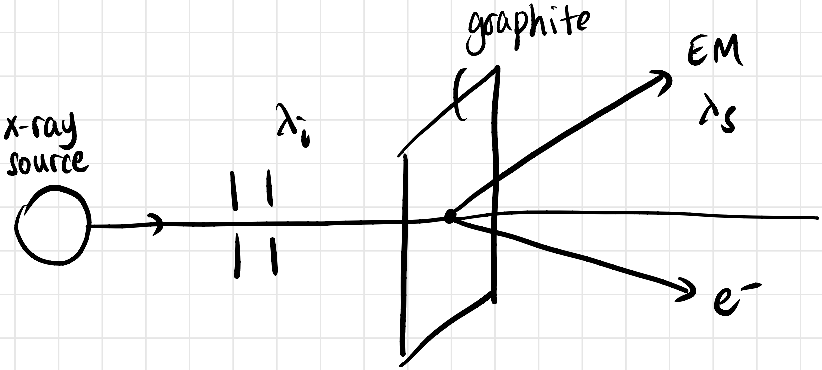
$$= \frac{8\pi\nu^3}{c^3} \partial\nu \, kT$$ — Rayleigh-Jeans Law

- at low frequencies, reduces to Rayleigh-Jeans Law

Planck's Law of blackbody radiation proves that radiation is a particle (discrete energy)

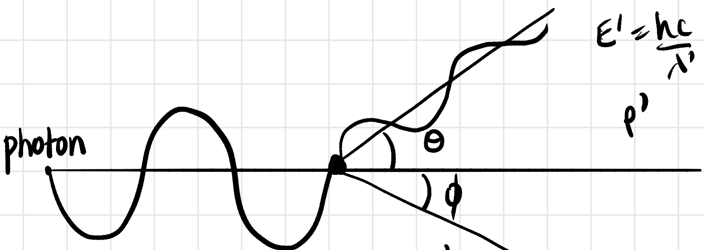
# Compton Effect

- Compton scattering
- Experiment that supported particle behaviour of EM radiation




$\lambda_s > \lambda_i$

Using particle picture of light (photons)




$E = h\nu$

$E = pc$

$p = \frac{h}{\lambda}$

$E_e = m_0c^2$

$p_e = 0$

$e^-$

$E_e, p_e$

# Law of conservation of energy

Total energy before collision = total energy after collision

$$E + m_0c^2 = E' + E_e \quad \text{——— (1)}$$

$$E_e = \sqrt{m_0^2c^4 + p_e^2c^2} \quad \text{——— Einstein's Theory}$$

(relativistic energy of moving particle)
$m_0$ = rest mass

# Law of conservation of momentum

## x-component of momentum

$$p + 0 = p'\cos\theta + p_e\cos\phi$$

$$p - p'\cos\theta = p_e\cos\phi \quad \text{——— (2)}$$

## y-component of momentum

$$0 = p'\sin\theta - p_e\sin\phi$$

$$p'\sin\theta = p_e\sin\phi \quad \text{——— (3)}$$

Squaring and adding (2) and (3)

$$(p - p'\cos\theta)^2 + (p'\sin\theta)^2 = (p_e\cos\phi)^2 + (p_e\sin\phi)^2$$

$$p^2 - 2pp'\cos 2\theta + p'^2 = p_e^2$$

$$p_e^2 = p^2 + p'^2 - 2pp'\cos\theta \longrightarrow (4)$$

Using equation (1)

$$(E - E') + m_0c^2 = E_e$$

$$E - E' + m_0c^2 = \sqrt{m_0^2c^4 + p_e^2c^2}$$

$$\left((pc - p'c) + m_0c^2\right)^2 = m_0^2c^4 + p_e^2c^2$$

$$(pc - p'c)^2 + \cancel{m_0^2c^4} + 2(pc - p'c)(m_0c^2) = \cancel{m_0^2c^4} + p_e^2c^2$$

$$p^2\cancel{c^2} + p'^2\cancel{c^2} - 2pp'\cancel{c^2} + 2c^3(p - p')m_0 = p_e^2\cancel{c^2}$$

Substituting $p^2 + p'^2$ from (4)

$$p^2 + p'^2 - 2pp' + 2(p - p')m_0c = p_e^2$$

$$\cancel{p_e^2} + 2pp'\cos\theta - 2pp' + 2(p - p')m_0c = \cancel{p_e^2}$$

$$2pp'(\cos\theta - 1) = 2(p' - p)m_0c$$

$$\frac{2h^2}{\lambda_i\lambda_s}(\cos\theta - 1) = 2\left(\frac{h}{\lambda_s} - \frac{h}{\lambda_i}\right)m_0c$$

$$2h\,(\cos\theta - 1) = 2(\lambda_i - \lambda_s)m_0c$$

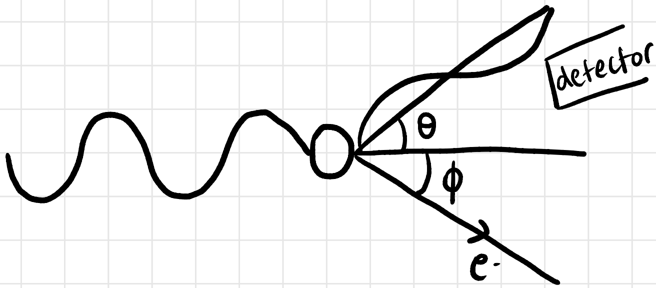
$\Delta\lambda = \lambda_s - \lambda_i$

$$\boxed{\lambda_s - \lambda_i = \frac{h}{m_0 c}(1-\cos\theta)}$$

Compton shift

$\Delta\lambda$ : Compton shift

$\frac{h}{m_0 c}$ : Compton wavelength $\approx 2.427 \times 10^{-12}$ m $= \lambda_c$




X-rays from Mo target ($\lambda = 0.074$ nm)

(i) $\theta = 0°$

$\Delta\lambda = 0$

photon not interacting with $e^-$

(ii) $\theta = 45°$

$\Delta\lambda = 0.71$ pm

(iii) $\theta = 90°$

$\Delta\lambda = \lambda_c = 2.427$ pm

(iv) $\theta = 180°$

$\Delta\lambda = 2\lambda_c = 4.854$ pm

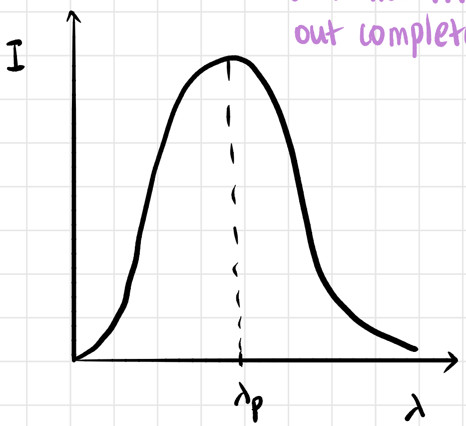
photons undergo backscattering

$e^-$ gains maximum energy

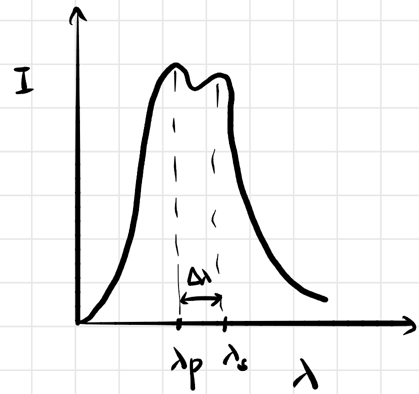
# I vs λ graph

(i) $\theta = 0°$

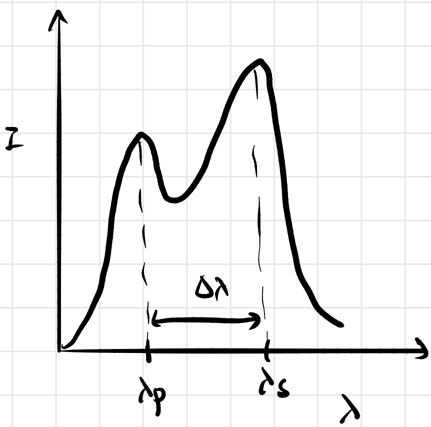
could not filter out completely



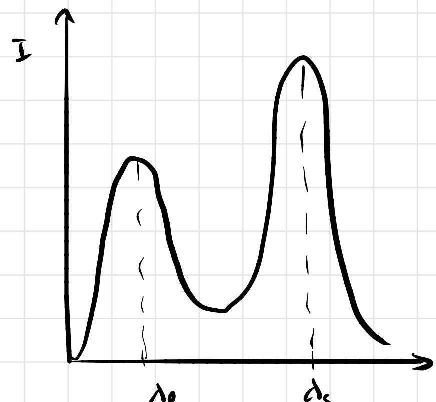

(ii) $\theta = 45°$




(iii) $\theta = 90°$




(iv) $\theta = 180°$




If a photon interacts with a tighter bound electron, the $e^-$ would not move / gain energy as it is in a stable state and photon would not lose energy and the wavelength stays $\lambda$.

This is because $\Delta\lambda = \dfrac{h}{m_{nucleus}\, c}(1-\cos\theta)$ and $m_{nucleus}$ is very high

photon with tight $e^-$

$\Delta\lambda \approx 0$

## Conclusion

- Compton shift does not depend on the incident wavelength
- Depends only on the scattering angle $\theta$

## de-Broglie Hypothesis

- Dual nature of matter
- Argued that if radiation shows dual nature, matter should too
- Every object in motion is associated with a wave, called matter waves

$$\lambda = \frac{h}{p}$$

- Cannot be observed for macroscopic objects as the momentum is large and the associated $\lambda$ is extremely small
- Only in atomic/subatomic scale

- Won Nobel prize in 1924
- Proven first by Davison-Germer experiment
- Used Ni crystal, $e^-$ was accelerated at different potentials

- $\lambda$ for a free particle

$$E_k = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE_k}$$

$$\lambda = \frac{h}{\sqrt{2mE_k}}$$

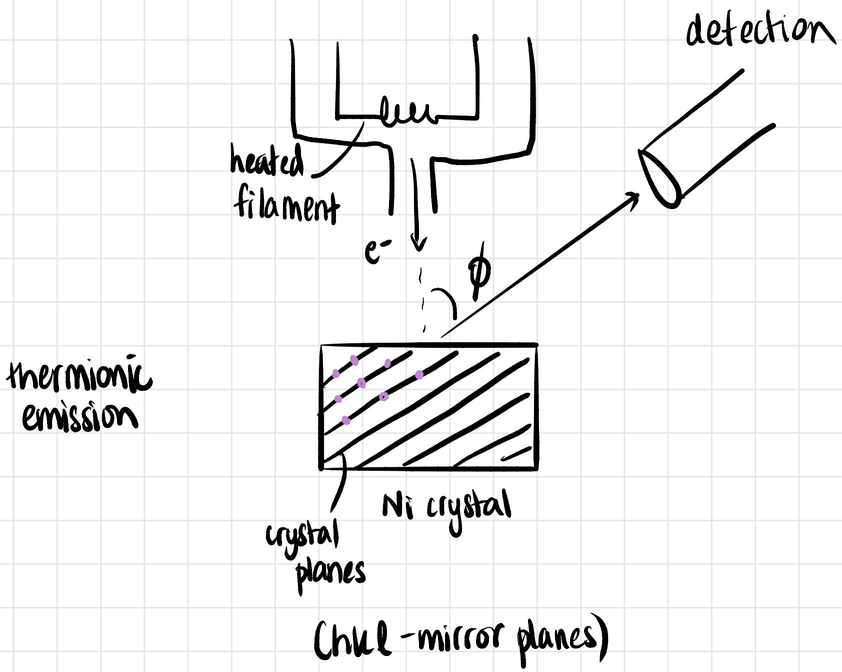
- For an accelerated charged particle

$$E_k = eV$$

$$\frac{1}{2}mv^2 = eV \Rightarrow \frac{p^2}{2m} = eV \Rightarrow p = \sqrt{2meV}$$

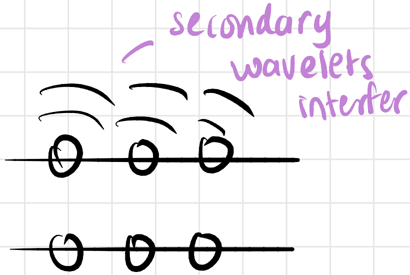
$$\lambda = \frac{h}{\sqrt{2meV}}$$

## Davisson-Germer Experiment




V=40
V=50
V=54
V=60

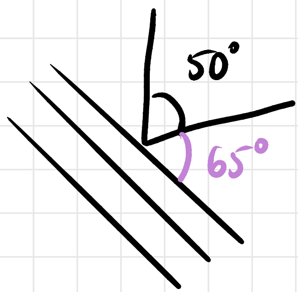
It was noticed that at V=54V and $\phi$ =50°, intensity was maximum




$2d\sin\theta = n\lambda$

Bragg's Law

- Interplanar distance known $d = \dfrac{a}{\sqrt{h^2+k^2+l^2}}$
- Using X-rays, they found lattice planes, miller indices found and interplanar space found to be $d = 0.91\,\text{Å}$
- For $e^-$, $2d\sin\theta = n\lambda$ (always take first order)




- Diffraction angle $= 65°$ (Bragg's diffraction)
- Use these values to calculate $\lambda \Rightarrow \lambda = 1.65\,\text{Å}$
- de Broglie $\lambda$ for accelerated charged particle

$$\lambda = \frac{h}{\sqrt{2eVm}} = 1.67\,\text{Å}$$

Q: Find the KE and $v$ of proton of mass $1.67 \times 10^{-27}$ kg associated with deBroglie wavelength of $0.2865$ Å

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

$$v = \frac{h}{m\lambda} = 13848 \ ms^{-1}$$

$$KE = \frac{1}{2}mv^2 = 1eV$$

Q: The shift in the $\lambda$ of x-rays scattered in a Compton experiment is $0.2pm$. $\lambda_s = 1.002 \, nm$. Find $\theta$ at which x-ray photon is scattered and what is the momentum gained by the $e^-$?

$$\Delta\lambda = 0.2 = \lambda_s - \lambda_i$$

$$\lambda_i = 1.0018$$

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

$$\cos\theta = 0.917$$

$$\theta = 23.42^\circ$$

$$\text{energy transferred} = \frac{hc}{\lambda_i} - \frac{hc}{\lambda_s}$$

homework

$$\frac{1}{2}\frac{p^2}{m} = hc\left(\frac{\Delta\lambda}{\lambda_i \lambda_s}\right)$$

$$p^2 = \frac{2mhc\,\Delta\lambda}{\lambda_i \lambda_s} = 2.685 \times 10^{-25}\ kg\,ms^{-1}$$

Ans: $6.6 \times 10^{-25}\ kg\,ms^{-1}$

Q: Compare the momentum and energy of $e^-$ and photon whose $\lambda_b = 650\,nm$

electron: $p = \frac{h}{\lambda}$

photon: $p = \frac{E}{c} = \frac{h}{\lambda}$

$$E = \frac{p^2}{2m}$$

$$E = \frac{hc}{\lambda}$$

$P_e = 1.019 \times 10^{-27}\ kg\,ms^{-1}$

$E_e = 3.565 \times 10^{-6}\ eV$

$P_p = 1.019 \times 10^{-27}\ kg\,ms^{-1}$

$E_p = 1.91\ eV$

$$\frac{E_e}{E_p} = 1.867 \times 10^{-6}$$

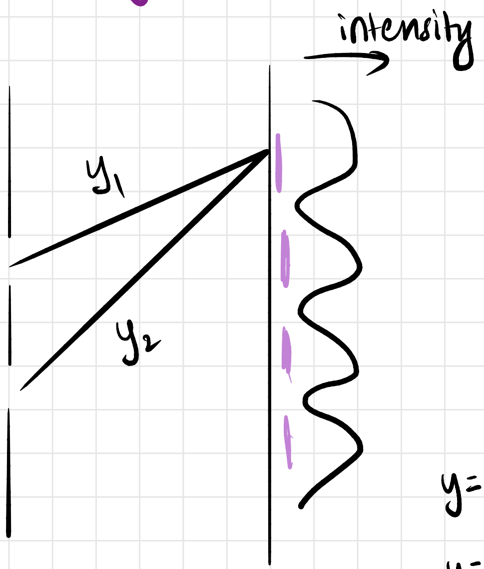
Q: What is $\lambda$ of H atom moving with the mean $v$ corresponding to the avg. KE of H atoms under thermal eq at 293 K (mass of H = 1.008 amu)

$$\frac{1}{2}mv^2 = \frac{3}{2}k_BT$$

$$\frac{p^2}{\cancel{2}m} = \frac{3}{\cancel{2}}kT \Rightarrow p = \sqrt{3kTm}$$

$$\lambda = \frac{h}{p} = 1.466\ \mathring{A}$$

## Interference (Light-revision)




$$y_1 = a\sin\omega t$$

$$y_2 = a\sin(\omega t+\phi)$$

$$y = y_1 + y_2$$

$$y = a(\sin\omega t + \sin(\omega t+\phi))$$

$$y = 2a\sin\left(\omega t+\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

$$I = y^2 = 4a^2\sin^2\left(\omega t+\frac{\phi}{2}\right)\cos^2\left(\frac{\phi}{2}\right)$$

$$\langle I \rangle = 4a^2 \left(\frac{1}{2}\right) \cos^2\left(\frac{\phi}{2}\right)$$

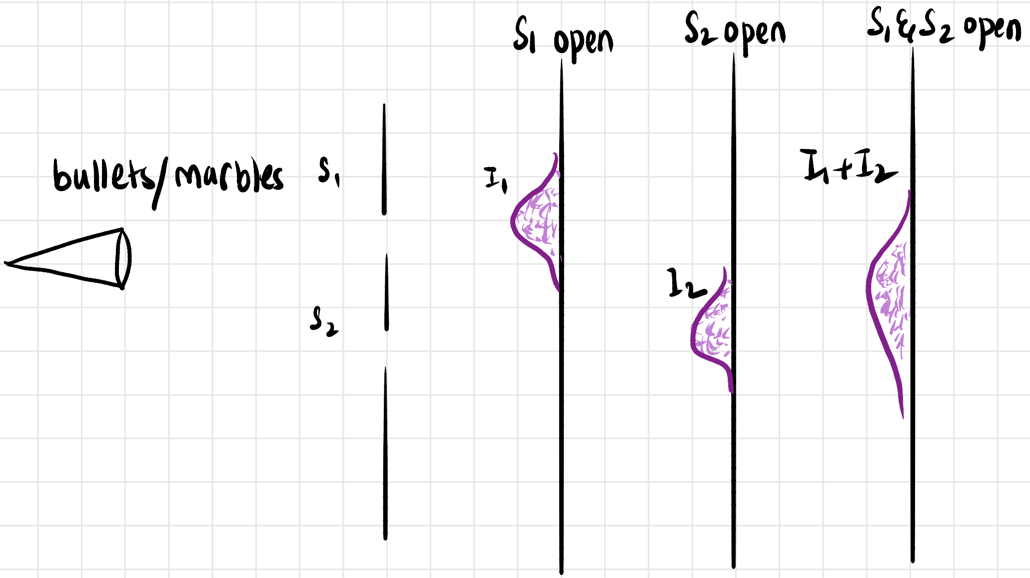
$$I \propto \cos^2 \frac{\phi}{2}$$

## Single Particle Double Slit Experiment

A single particle (photon, electron) can only go to one spot at a time.

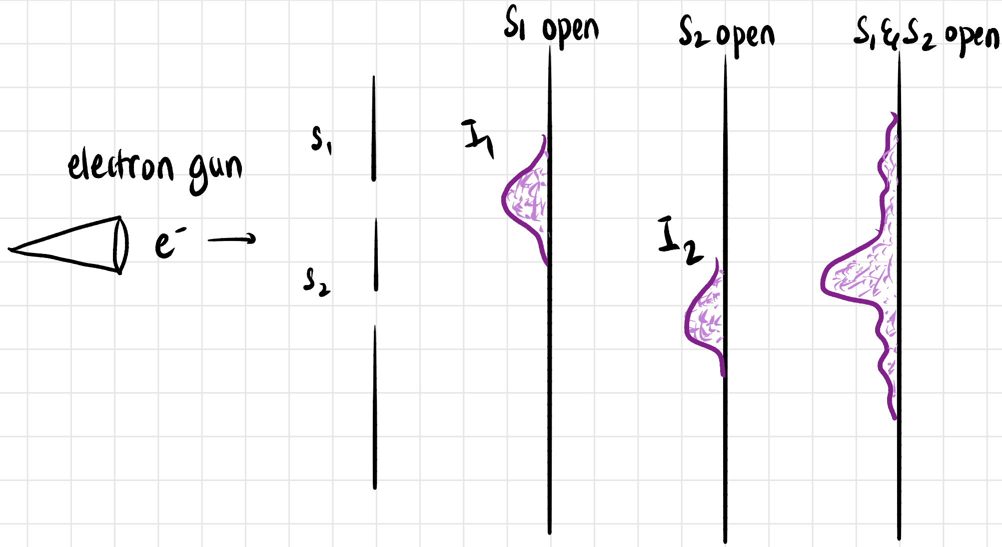
When one slit is open, we observe a normal distribution $I_1$ & $I_2$.

When both are opened, we expect to observe $I_{res} = I_1 + I_2$ as in the case of bullets. However, we notice an interference pattern. But this does not make sense as individual particles were sent one at a time (not light waves)



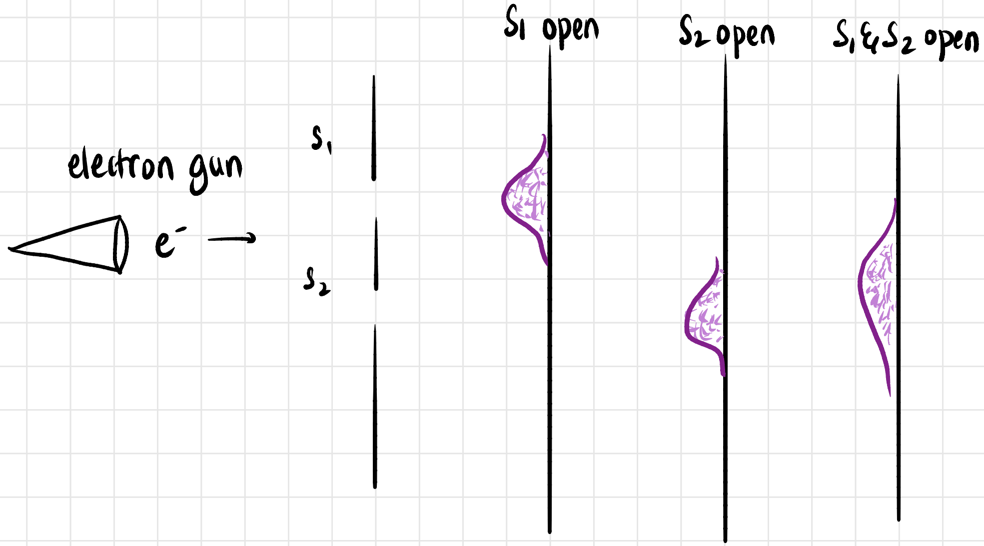
## Experimental setup





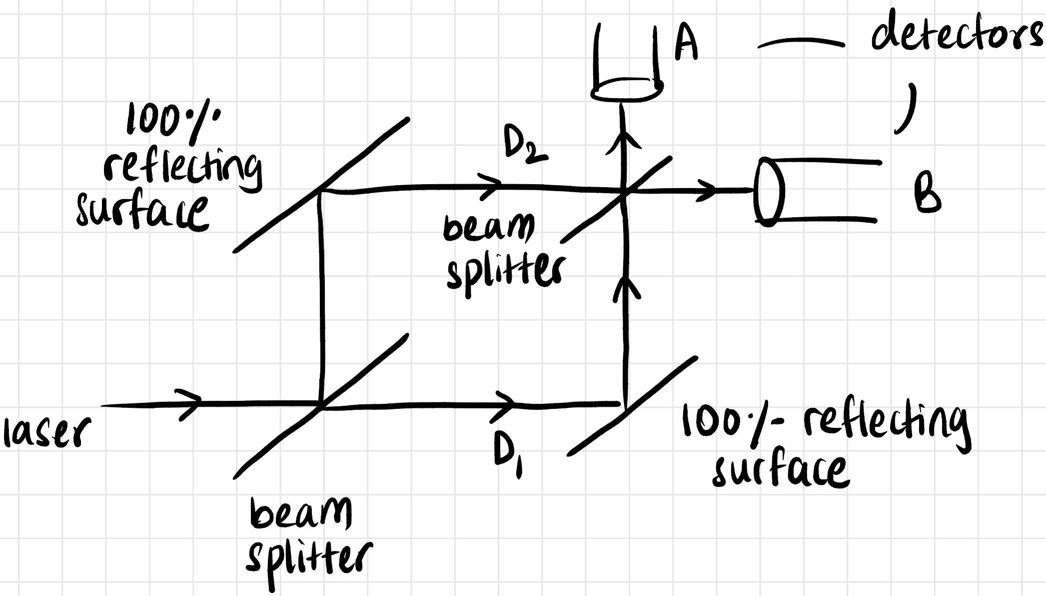

interference fringes
— wave nature

If detectors are placed, $\vec{E}$ disturbs wave nature of $e^-$ and particle behaviour observed.



# Mach Zehnder Experiment (interferometer)






Similar to single photon interference, a laser is shone as shown above.

## Beam splitter

splits beam into 50% intensity reflection, 50% intensity transmission.

When light is sent, all light reaches detector B and no light reaches detector A.

This is because the two paths to B result in constructive interference (in phase), while the two paths to A result in destructive interference (phase diff $= \pi$)

Paths to B: trans → refl → refl and refl − refl − trans

(0) ($\pi$) ($\pi$) ($\pi$) ($\pi$) (0)

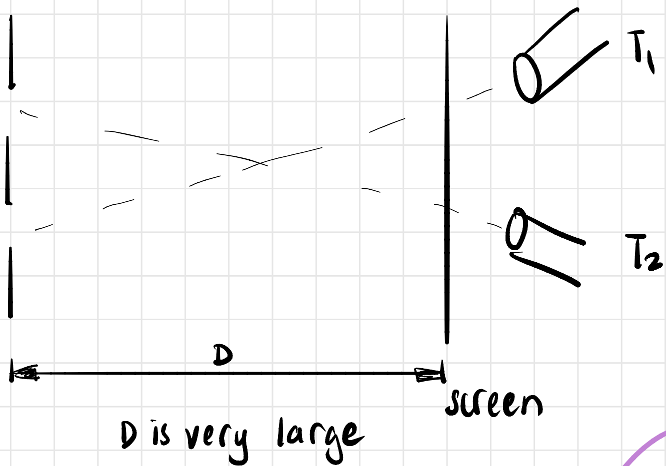
Paths to A: trans → refl → trans and refl − refl − refl

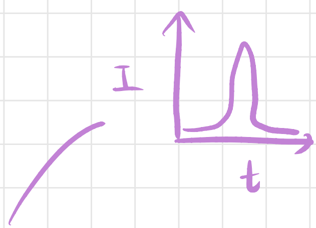
($\pi$) (0) (0) ($\pi$) ($\pi$) ($\pi$)

Even when performed with single particles, 100% of the particles go to B and 0% to A.

However, when detectors $D_1$ and $D_2$ are placed, 50% of the intensity is at A and 50% at B.

## John Wheeler's Delayed Choice Experiment



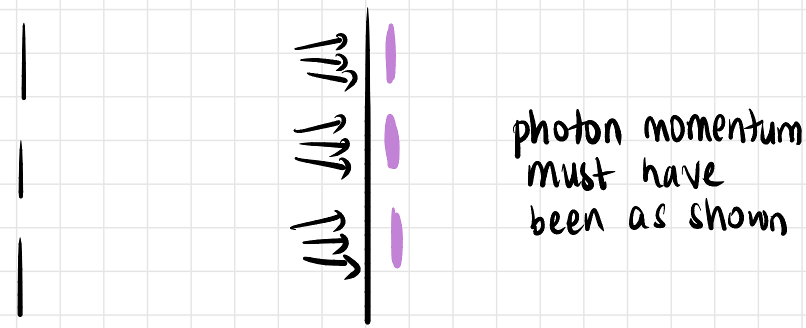

Send a short pulse of light (femtosecond) with many photons travelling towards the screen.

Two detectors $T_1$ and $T_2$ are placed behind the screen and are focused on each slit.

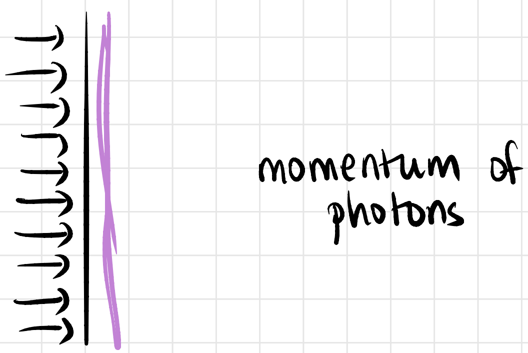
The screen can be made translucent in a fraction of a second by applying $\vec{E}$.

When experiment performed, we observe interference as usual when the screen is present.

This must mean that the photons were directed in such a way as to form the interference pattern.




If the screen is removed and the detectors are exposed, the interference pattern does not form. Instead, a continuous distribution of light is observed, which means photons were travelling like this.




How did the photons change their momentum to travel in all places instead of those certain area?

# Wavefunction ($\psi$ - psi)

de Broglie assumed that all matter has associated with it a wave, known as a matter wave (hypothetical)

This model accurately predicted experimental observations like interference, diffraction etc.

Waves signify variation of a certain parameter (E, pressure, water height)

EM waves — $\vec{E}$ & $\vec{B}$ — $E(x,t)$
strings — displacement — $y(x,t)$
sound wave — pressure — $P(x,t)$

In matter waves, what is varying?

Max Born assumed that all particles have associated with them a wavefunction with an associated wavelength.

$$\psi = A \sin\left(\frac{2\pi}{\lambda} x\right)$$

The wavefunction $\psi$ has no physical meaning; it is only a mathematical representation.

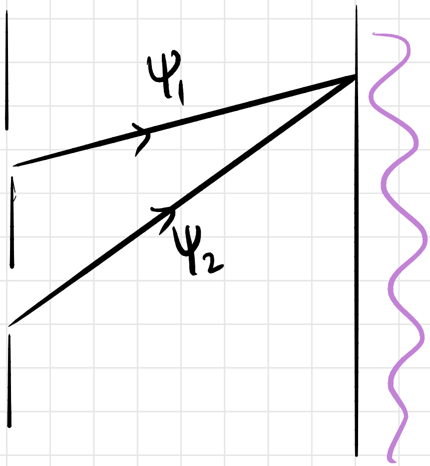
The wavefunction is defined as

$$\psi(x,t) = A \sin(\omega t - kx)$$

we assume a complex wavefunction for simplicity

$$\psi(x,t) = A e^{i(\omega t - kx)}$$ — purely mathematical

For interference,




$$I = |\psi_1 + \psi_2|^2$$

- We associate $\psi_1$, $\psi_2$ as wavefunctions for each slit (fictitious wave)

- Now, we notice that the intensity observed on the screen perfectly matches $|\psi_1 + \psi_2|^2$, and not $\psi_1^2 + \psi_2^2$.

- We imagine each particle sends 2 mathematical curves through both slits (the particle kind of 'splits')

- Fringe width $\beta = \dfrac{\lambda D}{d} = \dfrac{hD}{pd}$ ; depends on $\lambda$

- $\psi$ is purely mathematical (complex)

  $|\psi|^2$ is real (probability density)

- $e^-$ knows its surroundings and only goes to areas where the $\psi$ constructively interferes.

- $|\psi|^2 = \psi^* \psi$, where $\psi^*$ is complex conjugate
- eg: $\psi = Ae^{ikx}$, $\psi^* = Ae^{-ikx}$

$|\psi|^2 = A^2$ —— observable

## Probability density $|\psi|^2$

- Probability of finding the particle at a particular place when the space $\rightarrow 0$
- To find the probability of finding the particle in a finite place:

$$P = |\psi|^2 \Delta x \quad \text{or} \quad |\psi|^2 \Delta A \quad \text{or} \quad |\psi|^2 \Delta V$$

- $|\psi|^2$ becomes probability only when multiplied by some dimension.

$$dP = |\psi|^2 dx$$

- The probability that the particle lies in the region $a \leq x \leq b$ at any given time is given by

$$P_{a \leq x \leq b} = \int_a^b |\psi|^2 dx$$

- Total sum of probability = 1

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty} |\psi|^2 \, dz\, dy\, dx = 1$$

This is called normalisation

- $|\psi|^2$ is the probability per unit area/volume/distance
- $|\psi|^2$ can be greater than 1 as it is density
- But $P_i = |\psi|^2 dx < 1$ as $|\psi|^2 dx$ gives probability itself.

## Conditions on $\psi$

1. $\psi$ must be continuous everywhere (probability must be defined everywhere)

2. $\psi$ must be single-valued (single probability per point)

3. $\psi$ must be finite and as $x \to \infty$, $\psi \to 0$ (due to normalisation $\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} |\psi|^2 dx\, dy\, dz = 1$ and $\psi$ cannot be infinite)

4. $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$, $\frac{\partial \psi}{\partial z}$ and $\frac{\partial \psi}{\partial t}$ must be continuous everywhere.

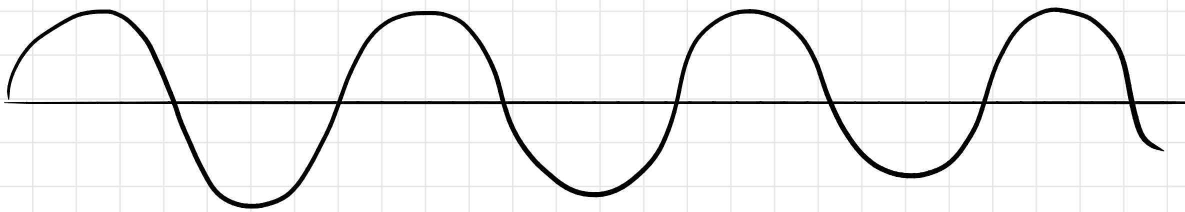
5. $\psi$ must be a solution to Schrödinger's Equation

6. $\psi$ must be normalisable

- In EM waves, $A^2 = I$
- In matter waves, $A^2$ gives probability density (probability of finding the particle)

## consider a sine wave

Amplitude same from $-\infty$ to $\infty$



To represent matter wave, we look for wave with varying amplitude

We superimpose many sine waves of slightly different frequencies and get wave packets.

Only a mathematical representation; not real

When many waves of slightly different frequencies are superimposed, the resultant is a wave packet / envelop

## Construction of Wave Packet

Superimpose waves with slightly different wavelengths.

For simplicity, we consider 2 waves and add/ superimpose them

# Phase & Group velocities

For any wave $y(x,t) = A\sin(\omega t - kx)$

$$v_{phase} = \frac{\omega}{k} = \frac{2\pi f\lambda}{2\pi} = f\lambda$$

phase velocity of the wave (how fast phase varies)

Consider two waves of slightly different frequencies

$$y_1 = A\sin(\omega t - kx) \quad \text{—— (1)}$$

$$y_2 = A\sin((\omega + \Delta\omega)t - (k + \Delta k)x) \quad \text{—— (2)}$$

Superimposing (1) and (2)

$$y = A\sin(\omega t - kx) + A\sin((\omega + \Delta\omega)t - (k + \Delta k)x)$$

$$y = 2A\sin\left(\omega t - kx + \frac{\Delta\omega t - \Delta k x}{2}\right)\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$

$$y = 2A\sin\left(\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$$

$$y = \underbrace{2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)}_{\text{amplitude}}\ \underbrace{\sin(\omega' t - k'x)}_{\text{phase}}$$

where $\omega' = \omega + \dfrac{\Delta\omega}{2}$

$k' = k + \dfrac{\Delta k}{2}$

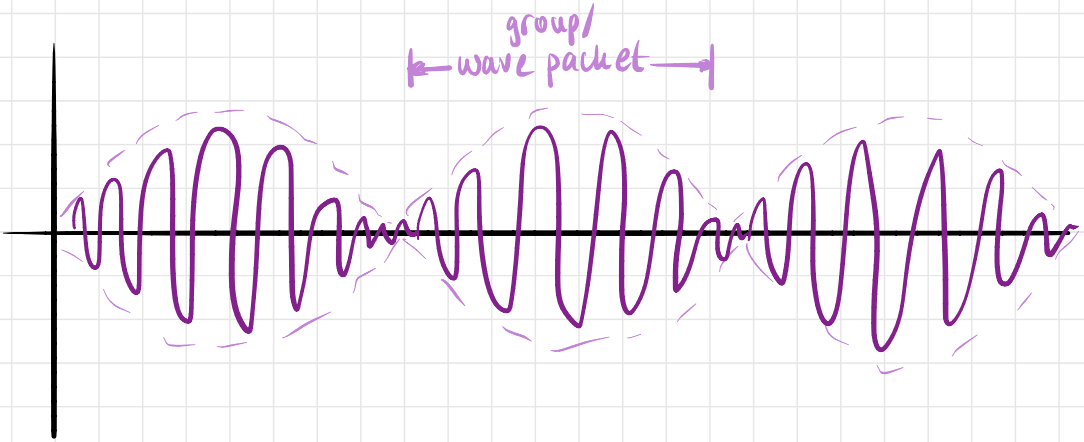
Amplitude varies with time (amplitude modulation)

There are two velocities in the wave.

Phase velocity
- actual velocity of the wave (how fast one phase moves)
- $\frac{\omega'}{k'} \approx \frac{\omega}{k} \longrightarrow$ high frequency component
- gives us momentum

Group velocity
- velocity of the wave packet / envelope / group
- $\frac{\Delta\omega}{\Delta k} \longrightarrow$ low frequency component
- gives us position

If $v_{phase} = v_{group}$, wave looks stationary; only horizontal movement is seen.




$$v_{group} = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

## Show that particle velocity = group velocity

$$v_{group} = \frac{d\omega}{dk} \qquad \omega = 2\pi f = \frac{2\pi E}{h}$$

$$\therefore v_{group} = \frac{dE}{dp} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

$$E = \frac{p^2}{2m} \Rightarrow \frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = v_{particle}$$

$$\therefore v_{particle} = v_{group}$$

In a dispersive medium, $v_{phase} = v_{group}$

In a non-dispersive medium, $v_{phase} \neq v_{group}$

## Relationship Between $v_{phase}$ and $v_{group}$

$$v_{phase} = \frac{\omega}{k} \qquad v_{group} = \frac{d\omega}{dk}$$

$$\omega = k\, v_{phase}$$

$$\frac{d\omega}{dk} = v_{phase} + k \frac{d v_{phase}}{dk}$$

$$v_{group} = v_{phase} + \frac{2\pi}{\lambda} \frac{d\,v_{phase}}{d\lambda} \frac{d\lambda}{dk}$$

$$k = \frac{2\pi}{\lambda} \Rightarrow dk = \frac{-2\pi}{\lambda^2}\, d\lambda$$

$$v_{group} = v_{phase} + \frac{2\pi}{\lambda}\left(\frac{-\lambda^2}{2\pi}\right) \frac{d\,v_{phase}}{d\lambda}$$

$$v_{group} = v_{phase} - \frac{\lambda\, d\,v_{phase}}{d\lambda}$$

Q: Evaluate the condition under which

1) $v_{group} = \frac{1}{2} v_{phase}$

$$v_g - v_p = -\lambda \frac{dv_{phase}}{d\lambda}$$

$$\frac{1}{2} v_p = \lambda \frac{dv_p}{d\lambda}$$

$$\int \frac{1}{2} \frac{d\lambda}{\lambda} = \int \frac{dv_p}{v_p}$$

$$\frac{1}{2} \ln \lambda = \ln v_p + C$$

$$v_p \propto \lambda^{1/2}$$

2) $v_{group} = 2 v_{phase}$

$$v_g - v_p = -\lambda \frac{dv_p}{d\lambda}$$

$$v_p = -\lambda \frac{dv_p}{d\lambda}$$

$$\int -\frac{d\lambda}{\lambda} = \int \frac{dv_p}{v_p}$$

$$-\ln \lambda = \ln v_p + C$$

$$v_p \propto \frac{1}{\lambda}$$

Q: Phase velocity of ripples on a liquid surface is $\sqrt{\frac{2\pi S}{\lambda\rho}}$ where S is the surface tension, $\rho$ is density.

Find $v_g$ in terms of $v_p$.

$$v_g = v_p - \lambda \frac{d v_p}{d\lambda}$$

$$v_g = \sqrt{\frac{2\pi S}{\lambda\rho}} - \lambda\sqrt{\frac{2\pi S}{\rho}}\left(\frac{-1}{2}\lambda^{-3/2}\right)$$

$$= v_p - \lambda^{3/2} v_p\left(\frac{-1}{2}\lambda^{-3/2}\right)$$

$$v_g = \frac{3}{2} v_p$$

Q: $v_p$ of ocean waves is $\sqrt{\frac{g\lambda}{2\pi}}$ where g = acc. due to gravity. Find $v_g$ in terms of $v_p$.

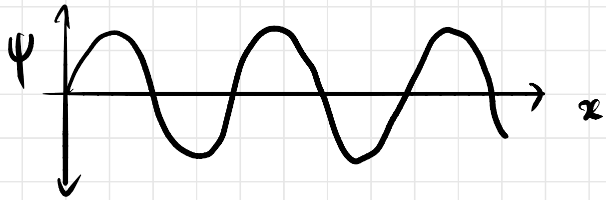
$$v_g = v_p - \lambda\sqrt{\frac{g}{2\pi}}\left(\frac{1}{2\sqrt{\lambda}}\right) = v_p - \frac{1}{2}v_p$$

$$v_g = \frac{1}{2} v_p$$

# HEISENBERG'S UNCERTAINITY PRINCIPLE

According to deBroglie, $\lambda = \frac{h}{mv}$ where $\lambda$ represents a wave

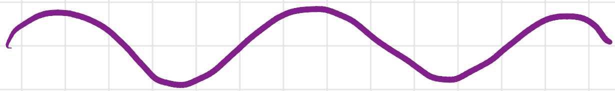
Let $\psi = Ae^{ikx}$ which is a wavefunction of a particle and we get $|\psi|^2 = A^2$



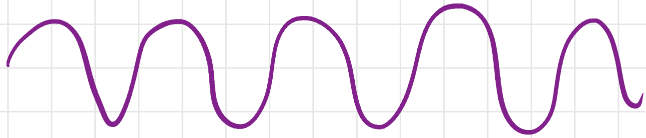


- We know $\lambda$ exactly $\Rightarrow$ p is exactly known
- $|\psi|^2 \rightarrow$ probability density is constant everywhere, which means the probability of finding the particle is constant everywhere. Therefore, the position of particle is unknown.
- To find position, we apply fourier transforms.
- We saw by adding two waves, we got packets, but those packets were everywhere.

- If we add multiple waves,

$\psi_1 - \omega_1, k_1$ 

$+$

$\psi_2 - \omega_2, k_2$ 

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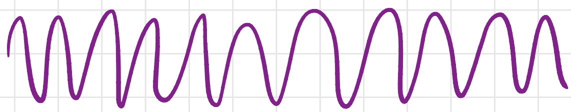
$\psi_3 - \omega_3, k_3$ 

$+$

$\vdots$

we get localised packets (packets only in one position)

$\psi =$ 

- If we make $\Delta\omega \longrightarrow$ very, very small
  $n \longrightarrow$ very, very large

(add many waves), we get

$\psi$ 

$x_0$

- We know the position of the particle fairly accurately, but since we added so many waves of different $\lambda$, the momentum of the particle is unknown.

- Fourier transform gives localised peak called as Dirac-Delta function

- If $\Delta x = 0$, $\Delta p = \infty$
  $\Delta p = 0$, $\Delta x = \infty$

## FOURIER INTEGRAL

- more on it later

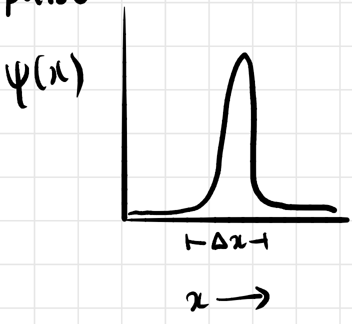
$$\psi(x) = \int_0^\infty g(k) \cos kx \, dk \longrightarrow \text{fourier integral}$$

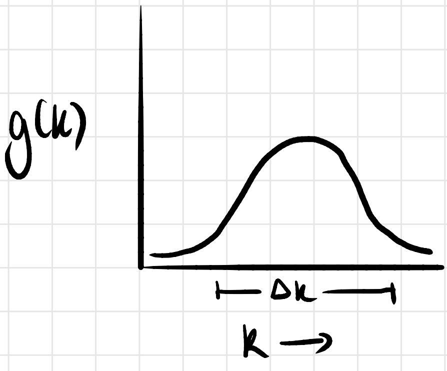
$g(k)$ ↑ amplitude

gives us wavefunction

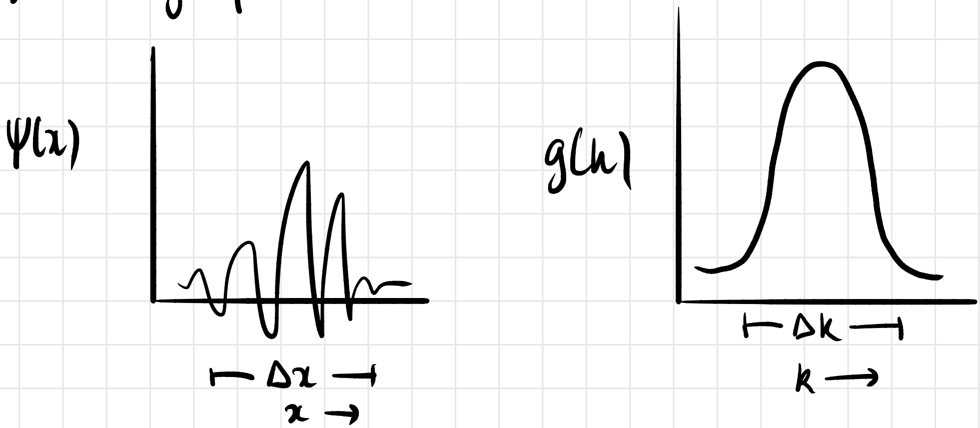
If we take various fourier integral waveforms

1) pulse

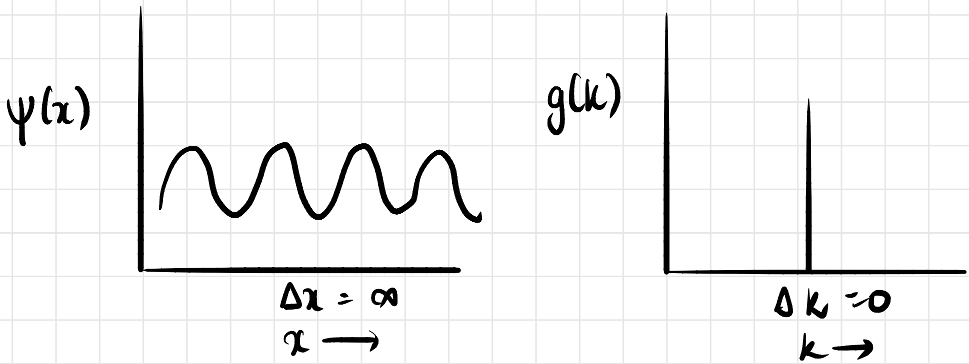


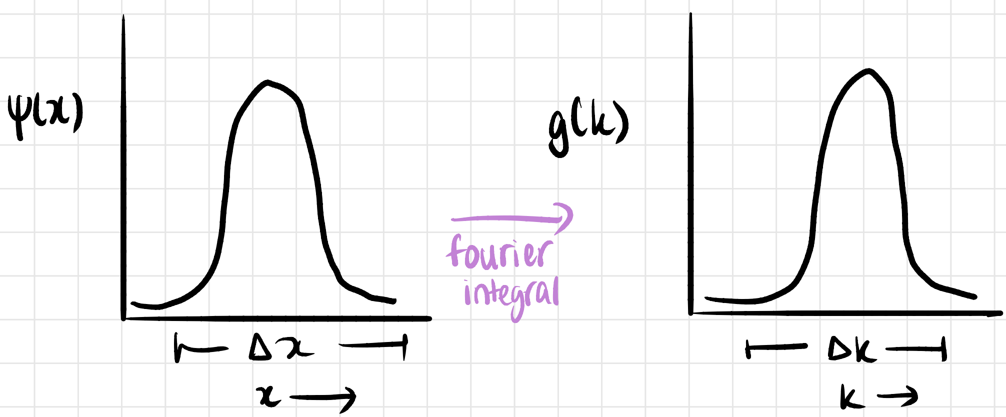
2) Wave group




3) wave train




4) Gaussian



The product of $\Delta x$ and $\Delta k$ is minimum for Gaussian wavepackets.

standard deviation of $\Delta x$ and $\Delta k$, as functions of $\psi(x)$ and $g(k)$, we get $\Delta x \Delta k = \frac{1}{2}$

Generally, wavepackets are not of Gaussian type

$$\Delta x \Delta k \geq \frac{1}{2}$$

$$k = \frac{2\pi}{h} p \Rightarrow \Delta k = \frac{2\pi}{h} \Delta p$$

$$\Delta x \Delta p \geq \frac{h}{4\pi} = \frac{\hbar}{2}$$

## Other Uncertainty relations

$$\Delta L \Delta \theta \geq \frac{h}{4\pi}$$ (angular)

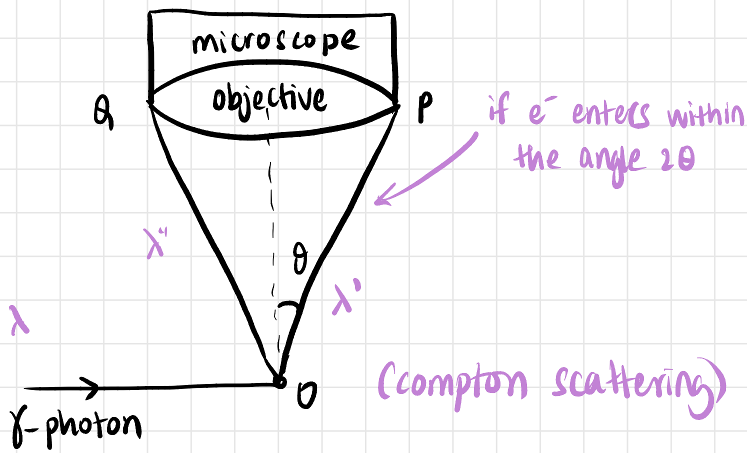
$$\Delta E \Delta t \geq \frac{h}{4\pi}$$ (energy)

Statement: It is impossible to measure momentum and position simultaneous with unlimited precision.

## Illustration of Uncertainty Principle

### Gamma Ray Microscope

- A thought experiment




- Limit to which position of $e^-$ can be measured is resolving power

$$\Delta x = \frac{\lambda}{2\sin\theta} \qquad \text{——— (1)}$$

- Using Compton scattering, find $\Delta p$

### Extreme cases

1) If photon enters eyepiece through OP ($p_{min}$)

- Momentum in x-direction

$$\frac{h}{\lambda} + 0 = \frac{h}{\lambda'} \sin\theta + p_{min}$$

2) If scattered photon enters through OQ ($p_{max}$)

$$\frac{h}{\lambda} + 0 = \frac{h}{\lambda''} \cos(90+\theta) + p_{max}$$

Uncertainty in momentum

Momentum can actually lie between $p_{min}$ and $p_{max}$

$$p_{max} - p_{min} = \frac{h}{\lambda'} \sin\theta + \frac{h}{\lambda''} \sin\theta = h\sin\theta\left(\frac{\lambda' + \lambda''}{\lambda'\lambda''}\right)$$

$\lambda \approx \lambda' \approx \lambda''$

$$\Delta p = \frac{2h \sin\theta}{\lambda}$$

$$\Delta p = \frac{2h}{\lambda} \sin\theta \quad \text{—— (2)}$$

From (1) and (2)

$$\Delta x \, \Delta p = \frac{\cancel{\lambda}}{\cancel{2\sin\theta}} \cdot \frac{\cancel{2}h \cancel{\sin\theta}}{\cancel{\lambda}} = h$$

$$\boxed{\Delta x \, \Delta p \simeq h}$$

Note: $h/4\pi$ comes from a different derivation involving standard deviation and fourier transforms.

Important: Here, we see $\Delta x$ & $\Delta p$ are limitations due to our measurement, but in reality these uncertainties are inherent to the particle itself.

# Nonexistence of $e^-$ Inside of Nucleus

- Let us assume $e^-$ exists inside nucleus
- If the $e^-$ is part of the nucleus, then the position of the $e^-$ is uncertain to the extent of the nuclear diameter.

$$\Delta x = D = 10^{-14} \text{ m}$$

- According to HUP, $\Delta x \Delta p \gtrsim \frac{h}{4\pi}$

$$\therefore \Delta p \approx 5.27 \times 10^{-21} \text{ kg m s}^{-1}$$

- We know from β-decay studies that the energy of the $e^-$ is about 3-4 MeV.
- We make an assumption that the momentum is of the order of the error
- The minimum momentum of the $e^-$ has to be the uncertainty $\Delta p$
- Therefore, $p = \Delta p$
- $E = \frac{p^2}{2m} \approx \frac{(\Delta p)^2}{2m} = 95.48 \text{ MeV}$
- The order of the energy of the $e^-$ we get is out of range of the energy of $e^-$
- Therefore, the $e^-$ cannot exist inside the nucleus.